Assignment 2

Date Due: Thursday, February 4 at 5:00

Reading: Chapter 6, pages 93-119 (Chapter called Congruence Classes)

Problems:

1. Find the following residues.
   i. \(5^{11} \pmod{7}\)
   ii. \((6^{44} + 13^{14}) \pmod{12}\)
   iii. \((140 \times 27) + (2^{19} + 1) + 83 \pmod{5}\)

2. Prove that if \(x \equiv y \pmod{m}\) then \((x,m) = (y,m)\).

3. Find all numbers \(b\) between 1500 and 1550 which are congruent to \(a\) mod \(n\) when:
   (1) \(a=1, n = 13\)
   (2) \(a=1515, n = 15\)

4. Prove that for all \(a, b\) and \(n\), \(ab \equiv (a \pmod{n}) \times (b \pmod{n}) \pmod{n}\)

5. Assume that \(a\) and \(b\) are positive integers and that \(a|b\).
   (i). Prove that \(\forall x \ (x \mod{b}) \mod{a} = x \mod{a}\)
   (ii). \(\forall x, y \ x \equiv y \pmod{b}\) implies \(x \equiv y \pmod{a}\)

6. Find all \(a \geq 0\) such that \(10a \equiv 0 \pmod{36}\). Justify your answer.

7. From all these rules about \(\mod\), it is tempting to make the following similar claim:
   \(\forall k, n \in \mathbb{N} \ a^k \equiv a^{(k \mod n)} \pmod{n}\)
   Is this true? Prove your answer.
8. Prove that \( \sum_{i=1}^{n-1} i = 0 \pmod{n} \) if \( n > 0 \) is odd.
(Hint: One way is to think about negative numbers mod \( n \).)

9. Define a sequence of natural numbers by, \( a_1 = 2 \), and for \( n > 1 \), \( a_{n+1} = a_n(a_n - 1) + 1 \).
Prove that \( a_n = a_{n-1}a_{n-2}...a_1 + 1 \) and that each of the \( a_i \)'s are relatively prime.

10. You are given 2 jars, one holds exactly 6 quarts and one holds exactly 11 quarts, and one bowl which holds more than 60 quarts. You want to put 13 quarts of water into the bowl using the jars. How do you do it?