CS 332 - Brief Answers to Homework 2

Due: Thursday, February 7

You should keep an extra copy all your homeworks if you do not typeset the answers.
To turn in HW 2 and all future homework you should use Gradescope. If needed Gradescope will
be reviewed next Monday in the lab sections. Just ask Jaiyu.

When you register for Gradescope please use your BU ID and also your full first and last name
that is on your BU records. Also, at the start of your paper you submit, write the class number
(CS 332) and also which homework assignment it is (HW 2) and write the number of the problem
at the top of your problem solution for each problem.

Reading: Chapter 3, section 3.2 and 3.3 of the Sipser textbook, pages 176-187 And chapter 4,
section 4.1, pages 194-201

PROBLEMS: Answer the following 6 questions.

1. For each of the following sets L, say whether they are languages and briefly say why or why
not.
   i. L = \{the binary string $b_0, b_1, b_2, \ldots | b_i = 1$ whenever the $i$ is a prime number \}

   ANS: Languages are (finite or infinite) sets of FINITE strings.
   L is an an infinite string of bits, hence NOT a language.

   ii. L= \{ the real number $\pi$ \}

   ANS: Not a language. $\pi$ when expressed as a real number is not a finite string.

   iii. L = \{2,3,4\}^*

   ANS: L is a language as it is an infinite set of all finite strings made up of 2’s, 3’s and 4’s.

2. Let A be any fixed finite set of 4 or more elements. Prove that the number of permutations
of elements in A is greater than the number of subsets of elements of A.

   ANS: This is an induction proof, using the fact that when $|A| = k$, the number of permutations
   of elements in A is $k!$ and that the number of subsets of A = $2^k$.

   (The assumption here is that students have had cs 311 and so know these facts.)

   So now prove that $K! > 2^k$ when K is at least 4.

   The base case is $k = 4$ where $4! = 24 > 2^4 = 16$.

   The induction step is form $k$ to $k+1$: if $k! > 2^k$ then $(k+1)! > 2^{k+1}$, which needs to be proved.
   This is a straightforward induction step.
3. Consider the statement \((\forall x \in S)(\exists y \in S)(\exists z \in S)\ (y \leq x \land z \leq x \land yz = x)\).

i. Is this statement true when \(S = N^+\)? \(S = N\)? When \(S = Z\)? When \(S = Q\)?

ii. Explain your answer for each question using no more than two sentences.

ANS for i. and ii:

When \(S = N^+\) the statement is true because for any \(x\) in \(N^+\) choose \(y = 1\) and \(z = x\).
When \(S = N\) the statement is true because for any \(x\) except 0 in \(N\) choose \(y = 1\) and \(z = x\), and for \(x=0\) choose \(y=0\) and \(z=0\).
When \(S = Z\) the statement is false because for \(x = -1\) there is no \(y\) and \(z\) in \(Z\) and \(\leq -1\) whose product is negative.
When \(S = Q\) the statement is false because for \(x = -1\) there is no \(y\) and \(z\) in \(Q\) and \(\leq -1\) whose product is negative.

iii. What statement is the negation of each of the first statement above?

ANS: The negation is \((\exists x \in S)(\forall y \in S)(\forall z \in S)\ (y > x \lor z > x \lor yz \neq x)\).

iv. Answer the 4 questions for the negation.

ANS: The answers here are F for N or \(N^+\) and T for Z and Q. The reason is that the negation of a statement has opposite truth values of the original statement.

4. Consider the following 2 step procedure P and explain why P is not a legitimate algorithm of the kind we discussed in class.

P: The input to P is three natural numbers \(a, b, c\).

Step 1. Try all possible assignments of natural numbers to \(a, b, c\) and for each of these possible settings test if \(a^3 + b^3 = c^3\)

Step 2. If the test you make of the equation in 1 is ever true, then P outputs true, if all of the tests you make in Step 1 are false then P outputs false.

ANS: P is not algorithmic because it takes infinitely many steps to do the test in step 1 for all \(a, b\) and \(c\) and Step 2 would never have the chance to decide T F in this case.

5. Answer problem 0.12 on page 27 of our textbook.

ANS: The statement that all horses have the same color is false and the induction step given is not correct. More specifically,

The proof is correct for the base case.

However the very first induction step, when \(k = 1\) is wrong.
In this step we are to prove that the claim is true for \(h=1\) and prove it for \(h=k+1=2\).
Now, you can see this is wrong when you think about the statement “Take any set H of 1+1=2 horses.” This set H may have 2 horses of different colors. So there should not be a proof that “all the horses of H must be the same color”. But where does the proof given go wrong?

The place the answer is it is wrong is in the very last statement of the induction a step.

We leave you to read this induction step carefully and see what is wrong in the last step of reasoning given in the text.

6. A k-clique in a graph G is complete subgraph of G with k vertices. (So for example, G has a 3-clique if it contains some triangle.)
   i. Give an example of a graph with a 4 clique but which doesn’t have a 5-clique.

   ANS: One example is a graph with 5 vertices where 4 of the vertices are $K_4$ the complete graph with 4 vertices, and the fifth vertex is connected to none of the 4 $K_4$ vertices (or to one of them if you prefer).

   ii. Describe an algorithm which takes as input a graph G and which decides if G has a 3-clique or not.

   ANS: The algorithm takes $G = (V,E)$ where $|V| = n$ as input and tests for each 3-tuple of 3 different vertices whether those 3 vertices form a 3-clique in G.

   So one such algorithm is:
   1. Set flag = 0.
   2. For i = 1 to n, j = 1 to n, k = 1 to n, if (i,j,k) are 3 different integers then test of $(v_i, v_j), (v_j, v_k), (v_k, v_i)$ are all in E, set flag = 1.
   3. If flag = 0, output “G has no 3-clique”, else output “G has a 3-clique”.

   What is the order of the (time) complexity of your algorithm?

   ANS: The time complexity of this algorithm is $O(n^3)$ as there are $n^3$ many triples (i,j,k) tested in the loop in step 2. Each test takes constantly many steps.