or input strings; b) \( M_1 \) generates an encoding for the \( i \)th Turing machine \( M_i \) using the procedure described in Section 5.6; c) \( M_1 \) simulates \( M_H \) to determine if \( M_i \) halts on \( w = x_i \); d) if \( M_H \) says that \( M_i \) does not halt, \( M_1 \) accepts \( w \); e) if \( M_H \) says that \( M_i \) does halt, \( M_1 \) simulates \( M_i \) on input string \( w \). \( M_1 \) rejects \( w \) if \( M_i \) accepts it and accepts \( w \) if \( M_i \) rejects it. Clearly, \( M_1 \) recognizes strings in \( \mathcal{L}_1 \), which contradicts the nature of \( \mathcal{L}_1 \). Thus, \( M_H \) cannot exist. \( \square \)

The second unsolvable problem we consider is the **empty tape acceptance problem**: given a Turing machine \( M \), we ask if we can tell whether it accepts the empty string. We reduce the halting problem to it. (See Fig. 5.13.)

\[
\mathcal{L}_{ET} = \{ \rho(M) \mid L(M) \text{ contains the empty string} \}
\]

**Theorem 5.8.2** The language \( \mathcal{L}_{ET} \) is not decidable.

**Proof** To show that \( \mathcal{L}_{ET} \) is not decidable, we assume that it is and derive a contradiction. The contradiction is produced by assuming the existence of a TM \( M_{ET} \) that decides \( \mathcal{L}_{ET} \) and then showing that this implies the existence of a TM \( M_H \) that decides \( \mathcal{H} = \text{HALT} \).

Given an encoding \( \rho(M) \) for an arbitrary TM \( M \) and an arbitrary input \( w \), the TM \( M_H \) constructs a TM \( T(M, w) \) that writes \( w \) on the tape when it is empty and then simulates \( M \) on \( w \), halting if \( M \) accepts it. Thus, \( T(M, w) \) accepts the empty tape if \( M \) halts on \( w \). \( M_H \) decides \( \mathcal{H} \) by constructing \( T(M, w) \) and passing it to \( M_{ET} \). (See Fig. 5.13.) The language accepted by \( T(M, w) \) includes the empty tape if and only if \( M \) halts on \( w \). Thus, \( M_H \) decides the halting problem, which as shown earlier cannot be decided. \( \square \)

**Note:** \( \rho(M) = \langle M \rangle \), \( M \in \mathcal{T} \).

*Figure 5.13* Schematic representation of the reduction from \( \mathcal{H} \) to \( \mathcal{L}_{ET} \).