CS 530 - Fall 2018
Homework 1 - Brief Answers

Due: Tuesday, September 18 - submit via Gradescope

Reading : Section 4.2, pages 75-82 and Sections 28.1 and 28.2 pages 813-830 of the textbook

Problems: Please limit your answer to the following problems to at most 2 pages each.

1. Strassen's method

Assume you are handed an algorithm which multiplies two 2 by 2 matrices and uses 7 integer multiplications and 18 integer additions to do this.

i. Show how to use this algorithm iteratively to multiply two 4 by 4 matrices. Calculate exactly how many integer multiplications and integer additions are done? You should show your work and should compare the answer you get to the number of multiplications and adds which would be done if you turn the 4 by 4 algorithm into block multiplication of 2 by 2 matrices.

Answer: Strassen takes 7 multiplications and 18 +'s to multiply two 2×2 matrices. So to multiply two 4×4 we use 7x7 = 49 mults and 18x7 = 126 +'s. (work not shown here.)

ii. In Section 4.2 of the textbook it is shown that Strassen's method uses only $O(n^{log7})$ multiplications when n is an even power of 2. (You need not prove this.)

Using this result describe how you can get the same $O(n^{log7})$ multiplications even if you start with $n\times n$ matrices with n not an even power of 2.

Answer: One idea is to let n' be the smallest even power of 2 bigger than n. Now simply pad (expand) the original matrices you are multiplying with 0's so that they become $n'\times n'$ matrices. While n' is larger that n, note that it is at most 2n.

So now we are doing Strassen's algorithm for $n'\times n'$ matrices as usual and we know this takes $O(n'^{log7})$ multiplications. But as $n' \leq 2n$ we have at most $O((2n)^{log7})$ multiplications which is still $O(n^{log7})$.

2. A permutation matrix P is an n by n Boolean matrix with exactly one 1 in each row and one 1 in each column.

P is called a permutation matrix because if you multiply P by any n by 1 column vector V then PV is a permutation of V.

Given a permutation matrix P:

i. Explain what permutation of 1,2,...,n the matrix P corresponds to. Is P invertible ? If so, explain how to construct the inverse of P.

Answer: One possible way to define the permutation which matrix P corresponds to is to say that for i and j from 1 to n, i correspond to j if i is the column in which 1 appears in row j of P. So for the identity P, each i corresponds to itself, whereas for P being the matrix with 1's on its anti-diagonal, i corresponds to n-i+1. Note that here, 2 different P's give 2 different permutations and every permutation corresponds to exactly one P.

P is invertible and P^t (P transpose) is its inverse.

ii. What are the possible values of the determinant of P ? What is the relationship between the determinant of P and sign(Q), where Q is the permutation which corresponds to the matrix P ? Here sign(Q) is the number of flips (i.e., interchanges of permuted integers) that need to be done to go from the identity permutation to Q. Explain why the relationship holds.

Answer:
P has determinant equal to the its sign, so it is either 1 or -1, and it is 1 if and only the sign is even.

iii. Let P =

```
0  1  0  0
0  0  0  1
1  0  0  0
0  0  1  0
```

Construct the inverse of P and compute P's determinant.

Ans: P's inverse is its transpose and P's determinant is -1 since it takes an odd number of interchanges to go from the identity permutation to the permutation corresponding to P.

iv. Give an example of a 3 by 3 permutation P which has no LU decomposition.
See 3(i). below.

3. i). Prove that for any n greater than or equal to 2 there is a non-singular square (n by n) matrix which has no LU decomposition with L unit lower triangular and U upper triangular.

Answer:

```
                     0 0 0 1
                     0 1 0 0
Take for n = 4, M =  0 0 1 0
                     1 0 0 0
```

Then M is non singular but if there was a l and a U, we'd be able to prove they do not work (by showing $M_{5,1}$ is not 1 but 0.

Give a specific example of your matrix when n=3 and show its LUP decomposition.

Ans:

```
                     0 0 1
For M = 3x3 we have  0 1 0
                     1 0 0
```

and the L,U,P are ID, ID and M.

(ii). Give an example of a singular 2 by 2 matrix A and a singular 3 by 3 matrix B which have an LU decomposition. Both A and B should be non-zero matrices and you should write the L and the U for both of them.

Answer:

```
2x2 is 0 1
       0 1

M=3x3 is   0 0 1
           0 0 1
           0 0 1
```

and L=ID and U=M.