Problem Set #4

Due: Due Tuesday, November 19

For this and future assignments, please start each problem on a separate sheet of paper, and put your name on each sheet of paper.

Reading: Do the reading for the lectures next week.

Problems:

The first 3 problems all refer to the algorithm and proof given in class that a graph G with n nodes and average degree D has an independent set of size at least n/(D+1).

1. Prove that the bound on the size of independent sets is tight. That is, for any integer t, find a graph G of n vertices, where n ≥ t, where the size of the largest independent set in G has size exactly n/(D+1).

2. Given a graph G and vertex v in G, let dv be the degree of v. In the proof we claim that for any G, Σv∈V 1/(D + 1) ≤ Σv∈V 1/(dv + 1). Prove this.

3. In the algorithm you use a permutation of the n vertices of G chosen uniformly at random. Consider the following method to do this.
   i. Let V = (v1, v2, ..., vn) be a list of the n vertices in V.
   ii. For k = 1 to n
       Choose an integer i uniformly at random from the set {1,2,3,....n}. Switch vk with vi in the list.
   iii. Output the list you have after the loop in step ii. is executed.

   Question: Is the permutation that is output in step 3 a random permutation of V ? Why or why not? Note: What you need to determine here is whether all of the permutations of V are equally likely to be output in step 3 of the algorithm.

4. The 2-SAT algorithm on page 157 of the textbook was discussed in class.
   (i). Is this algorithm a Monte Carlo algorithm? Why or why not?
   (ii). Is this algorithm a Las Vegas algorithm? Why or why not?

   (For both parts above you should consider the results 7.1 and 7.2 stated on page 159 of the textbook.)

5. Consider the Markov chain defined by in exercise 7.1 on page 182 of the text.
   (i). Draw the graph picture of this chain, and compute P^2.
   (ii). Is this Markov chain irreducible? Why or why not. (See page 164)
   (iii). Find one recurrent state of the chain and say why it is recurrent.
   (iv). IS the recurrent state you found in (iii) periodic or aperiodic?
   (v). Find the stationary distribution of the Markov chain.

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(vi). How many communicating classes does this Markov chain have? What are they.

6. (i). Give an example of a Markov chain of at least 3 states where every state is recurrent. Justify your answer.
(ii). Give an example of a Markov chain of at least 3 states where some state is transient. Justify your answer.