Problem Set #5

Due: Due Thursday, December 12
Problems: Do any 2 out of the 4 problems.
For this assignment please start each problem on a separate sheet of paper, and put your name on each sheet of paper.

1. You are given a graph $G = (V, E)$ and a MST $T$ of $G$. Now the weight of one edge $e$ is changed from the original weight $w(e)$ to the new weight $w'(e)$. Denote the new graph as $G'$, and $T'$ as the new MST of $G'$. Give linear time algorithm to compute $T'$ for each of the following cases, and explain briefly why it works.
   1. $e \in T$ and $w'(e) < w(e)$
   2. $e \in T$ and $w'(e) > w(e)$
   3. $e \notin T$ and $w'(e) < w(e)$

2. The stable marriage problem does not handles divorces. This is because we assume everyone is interested in everyone else of the opposite sex and we assume that the preferences do not change.
   In this problem, we will see the effect of changes in preferences in the outcome of the Gale-Shapley algorithm (for this problem you can assume the version of the Gale-Shapley algorithm where the women do all the proposing).
   Given an instance of the stable marriage problem (i.e. set of men $M$ and the set of women $W$ along with their preference lists: $L_m$ and $L_w$ for every $m \in M$ and $w \in W$ respectively), call a man $m$ a home-wrecker if the following property holds. There exists a new list $L'$ for $m$ such that if $m$ changes his preference list to $L'$ (instead of $L_m$) then the Gale-Shapley algorithm matches everyone to someone else.
   In other words, let $S_{orig}$ be the stable marriage output by the Gale-Shapley algorithm for the original input and $S_{new}$ be the stable marriage output by the Gale-Shapley algorithm for the new instance of the problem where m’s preference list is replaced by $L'$ (but everyone else has the same preference list as before). Then $S_{orig} \cap S_{new} = \emptyset$.
   Prove: There exists an instance of the stable marriage problem with 4 men and 4 women such that there is a man who is a home-wrecker.

3. Recall that Carmichael numbers are those that satisfy Fermat’s theorem even when the number $n$ is prime.
   That is, for any $n$, define $D_n = \{x \in \mathbb{Z}_n^* | x^{n-1} = 1 (mod\ n)\}$.
   Then a Carmichael number is a composite $n$ where $D_n = \mathbb{Z}_n^*$.
   When $n$ is not a Carmichael number and $n$ is not prime then $|D_n| \leq (1/2)|\mathbb{Z}_n^*|$.  
   (i). Prove this last statement. (Hint: Use Lagrange’s theorem.)
   (ii). Use this last statement to give a simple randomized algorithm for primality which works whenever $n$ is not a Carmichael number. (Unfortunately there are infinitely many Carmichael numbers.)

4. For problem 4, see part 2 of the homework from the web page.