Random Routing Algorithm for Hybercube Routing

> **Two-Phase Routing Algorithm**:
>
> **Phase I** – Route the packet to a randomly chosen node in the network using the bit-fixing route.
>
> **Phase II** – Route the packet from its random location to its final destination using the bit-fixing route.

Algorithm 4.2: Two-phase routing algorithm.

What if MGF?

Definition: The moment generating function (MGF) of a random variable X is defined as

$M_X(t) = E[e^{tX}]$ Where t is a small neighborhood of zero.

- We will be interested mainly in the properties of this function around t=0.
- The MGF is a special function that " captures " all the moments of X
- Mainly interested in existence & properties of MGF.

Properties of MGF?

1) Let X be a Random Variable with MGF $M_X(t)$.

$M_X^{(n)}(0) = E(X^n)$ for all $n>1$, where $M_X^{(n)}(0)$ is the n^{th} derivative of $M_X(t)$ evaluated at t=0.

Proof: The assumption that expectation and differentiation operands can be exchanged;

$$M_X^{(1)}(t) = \frac{d}{dt}(E[e^{tX}]) = E[\frac{d}{dt}(e^{tX})]$$
$$= E[Xe^{tX}]$$
$$M_X^{(2)}(t) = \frac{d}{dt}(E[Xe^{tX}]) = E[\frac{d}{dt}(Xe^{tX})]$$
$$= E[X^2e^{tX}]$$
$$\dots$$
$$M_X^{(n)}(t) = E[X^ne^{tX}] \Rightarrow M_X^{(n)}(0) = E[X^n]$$

2) Let X and Y be two random variables. If there exists d>0 such that

$M_X(t) = M_Y(t)$ for all t in [- d,d], then X and Y have the same distribution.

3) If X and *Y* are independent RV then,

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

Proof:

$$M_{X+Y}(t) = E[e^{t(X+Y)}]$$
$$= E[e^{tX} e^{tY}]$$
$$= E[e^{tX}]\, E[e^{tY}]$$
$$= M_X(t)\, M_Y(t)$$

X & Y are independent then e^{tx} & e^{ty} are independent (line2 ->3).

Chernoff Bounds

The Chernoff bound for a RV X is obtained by applying Markov's in-equality to e^{tX} for some well-chosen value t. From Markov's inequality, we can derive the following useful inequality: for any $t > 0$.

$$\Pr(X \geq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

In particular,

$$\Pr(X \geq a) \leq \min_{t>0} \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

Similarly, for any $t < 0$,

$$\Pr(X \leq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

Hence

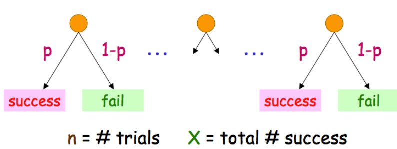
$$\Pr(X \leq a) \leq \min_{t<0} \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

We should choose an appropriate value for "t". While the value of t that minimizes $E[e^{tX}] / e^{ta}$ gives the best possible bounds.

Chernoff bounds for the Sum of Poisson Trials

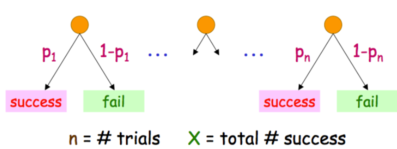
Now develop the most commonly used version of the Chernoff bound. We apply Chernoff bounds to bound the tail and the head distributions of sum of Poisson trials.

Bernoulli trials are a special case of Poisson trials. Bernoulli trials are random experiments. Has exactly 2 outcomes -> success / failure. Probability of success ->"1" is the same every time in the experience.



n independent indicators, each with success probability p. Binomial distribution gives the number of success in n independent Bernoulli trials. X = Bin(n,p)

In the ***Poisson trials*** each of indicators to choose its own success probability;



Let XI, ... , Xn be a sequence of independent Poisson trials with $\Pr(X_i = I) = P$

Let x = x1 +x2 +..+xn. Then;

μ = E[X] = E [X1+…+Xn] = P1+….+Pn

$$\mu = \mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} p_i.$$

We want use Chernoff bound to bound the tail distribution. For a given $\delta > 0$, we are interested in bounds on $\Pr(X \geq (1+\delta)/\mu)$ and $\Pr(X \leq (1-\delta)/\mu))$ - that is, the probability that X deviates from its expectation μ by $\delta\mu$ or more. We try to bound the total amount of probability of some random variable X that is in the "tail" far from the mean. A typical Chernoff bound concerns the probability that Y exceeds $1+\delta$ times its expectation μ. So now lets bound the sum of Poisson trials with $\Pr(X \geq (1+\delta)/\mu)$.

We start with the moment generating function of each X_i

M_{X_i} be the MGF of X_i.

$$\begin{aligned} M_{X_i}(t) &= E[e^{tX_i}] \\ &= p_i e^{t(1)} + (1-p_i) e^{t(0)} \\ &= p_i e^t + (1-p_i) \\ &= 1 + p_i (e^t-1) \\ &\leq e^{p_i (e^t-1)} \end{aligned}$$

In the last inequality we have used the fact that for any y, $1+y < e^y$. We take the product of n generating functions to obtain:

$$\begin{aligned} M_X(t) &= \Pi_i M_{X_i}(t) \\ &\leq \Pi_i e^{p_i (e^t-1)} \\ &= \exp\{\Sigma_i p_i(e^t-1)\} \\ &= e^{\mu(e^t-1)} \end{aligned}$$

So now we can develop concrete version of Cherrnoff bound for sum of Possion trials:

Theorem 4.4: *Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i) = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[X]$. Then the following Chernoff bounds hold:*

1. *for any $\delta > 0$,*

$$\Pr(X \geq (1+\delta)\mu) < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu; \tag{4.1}$$

2. *for $0 < \delta \leq 1$,*

$$\Pr(X \geq (1+\delta)\mu) \leq e^{-\mu\delta^2/3}; \tag{4.2}$$

3. *for $R \geq 6\mu$,*

$$\Pr(X \geq R) \leq 2^{-R}. \tag{4.3}$$

The first bound of the theorem is the strongest bound, the other 2 bounds are weaker than 1st one however they are easier to state and compute with in many situation.

Lets first proof the strongest bound: We are applying the Markov's inequality for any t>0 so we have:

Proof: For any $t > 0$, we have

$$\begin{aligned} &\Pr(X \geq (1+\delta)\mu) \\ &= \Pr(e^{tX} \geq e^{t(1+\delta)\mu}) \\ &\leq E[e^{tX}] / e^{t(1+\delta)\mu} \qquad \text{... (by which inequality?)} \\ &= M_X(t) / e^{t(1+\delta)\mu} \\ &\leq e^{\mu(e^t-1)} / e^{t(1+\delta)\mu} \end{aligned}$$

To get the best bound for $\Pr(X \geq (1+\delta)\mu)$, we now choose t so as to minimize the term

$$e^{\mu(e^t-1)} / e^{t(1+\delta)\mu}$$

One important question is here which t should we choose?

Observation:

To minimize $e^{\mu(e^t-1)} / e^{t(1+\delta)\mu}$

$\Leftrightarrow$ to minimize $\log_e (e^{\mu(e^t-1)} / e^{t(1+\delta)\mu})$

So, we want to choose t so as to minimize the term $\log_e (e^{\mu(e^t-1)} / e^{t(1+\delta)\mu})$, which is

$$\mu(e^t-1) - t(1+\delta)\mu$$

By calculus, the best t will be $\log_e (1+\delta)$

So, by substituting $t = \log_e (1+\delta)$ in the previous inequality:

$$\Pr(X \geq (1+\delta)\mu) \leq e^{\mu(e^t-1)} / e^{t(1+\delta)\mu}$$

we get:

$$\Pr(X \geq (1+\delta)\mu) \leq e^{\mu(\delta)} / (1+\delta)^{(1+\delta)\mu} = (e^{\delta}/(1+\delta)^{(1+\delta)})^{\mu}$$

Proof of Weakers Bound:

To obtain (4.2) we need to show that, for $0 < \delta \leq 1$,

$$\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \leq e^{-\delta^2/3}.$$

$$\delta - (1+\delta) \log_e (1+\delta) \leq -\delta^2/3$$

(this is obtained by taking log on both sides)

Let $f(\delta) = \delta - (1+\delta) \log_e (1+\delta) + \delta^2/3$

Target: to show $f(\delta) \leq 0$

Then, $f'(\delta) = 1 - (1+\delta)/(1+\delta) - \log_e (1+\delta) + 2\delta/3$

$= -\log_e (1+\delta) + 2\delta/3$

and

$f''(\delta) = -1/(1+\delta) + 2/3$

So, we see that

$f''(\delta) < 0$ for $0 \leq \delta < 1/2$

$f''(\delta) \geq 0$ for $1/2 \leq \delta \leq 1$

Example:

We will now use Chernoff bounds to analyze n coin flips of an unbiased coin.

Let $X_i = 1$ if the i^{th} flip is heads and 0 otherwise. Let $X = \sum X_i$ be the number of heads in n flips. It is immediate that we expect to see $\mu = n/2$ heads. We now compute the deviation using Chernoff bounds.

Let X = # heads in n fair coin flips

By Markov: $\Pr(X \geq 3n/4) \leq 2/3$

By Chebyshev: $\Pr(X \geq 3n/4) \leq 4/n$

By Chernoff:

$$\Pr(X \geq 3n/4) = \Pr(X \geq (1.5)\mu) \leq e^{-\mu(0.5)^2/3} = e^{-n/24}$$

References:

1) Probability and Computing: Randomized Algorithms and Probabilistic Analysis by Michael Mitzenmacher and Eli Upfal, pg 61-78.