Stable Roommates Problem

Each person in a set of even cardinality n ranks the others in order of preference. A *matching* is a partition of the set into $\frac{n}{2}$ pairs. A matching is called *stable* if no two persons (who are not paired) both prefer each other to their actual partners.

Contrary to the case of Stable Marriage problem, stable matching doesn't always exist.

Stable Roommates Algorithm (Irving, 1985)

Each person remembers his *favorite* - the person who gave him the best proposal.
If X receives a proposal from Y, X compares Y with his current favorite F. If F is more preferrable then Y, then X *rejects* Y's proposal. Otherwise X accepts Y's proposal, rejects F and takes Y as a new favorite.

The first phase of the algorithm:

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for person from 1 to N do {
    proposer = person
    repeat {
        proposer proposes to the next candidate in his list
        if proposer is not rejected then {
              candidate's favorite = proposer
              proposer = candidate's previous favorite
        }
    }  until proposer is empty or no more candidates in the list
}
```

Example

1	4	6	2	5	3
2	6	3	5	1	4
3	4	5	1	6	2
4	2	6	5	1	3
5	4	2	3	6	1
6	5	1	4	2	3

$1 \to 4$; 4's favorite is 1
$2 \to 6$; 6's favorite is 2
$3 \to 4$; 4 rejects
$3 \to 5$; 5's favorite is 3
$4 \to 2$; 2's favorite is 4
$5 \to 4$; 4's favorite is 5; rejects 1
$1 \to 6$; 6's favorite is 1; rejects 2
$2 \to 3$; 3's favorite is 2
$6 \to 5$; 5 rejects
$6 \to 1$; 1's favorite is 6

If after the first phase there is a person rejected by everyone, then there is no stable matching. Otherwise each person has a favorite, and we can reduce the table. Assume Y's favorite is X. Then you can remove from Y's list:

(i) all those to whom Y prefers X (*)
(ii) those who's favorite is the person whom they prefer to Y (+).

1	4+	6	2*	5*	3*
2	6+	3	5	1+	4
3	4+	5	1+	6+	2
4	2	6+	5	1*	3*
5	4	2	3	6*	1*
6	5+	1	4*	2*	3*

Reduced table:

1	6		
2	3	5	4
3	5	2	
4	2	5	
5	4	2	3
6	1		

If every list now contains just one person, then it is a stable matching.

All-or-nothing cycle is a cyclic sequence $a_1, \dots, a_r$, where the second person in a_i's list (b_{i+1}) is the first in a_{i+1}. To reduce the cycle means to delete b_i from a_i's list and to delete all successors of a_i in b_{i+1}'s list.

The second phase of the algorithm:

Find cycles and reduce them.

If there is an empty list, then there is no matching. If each list contains only one candidate, then they form a stable matching.

Expected number of stable matchings.

Assume you have a random ranking matrix. How many stable matchings are there?

Suprisingly, we can reduce this combinatorial problem to calculus. First, let us describe how we will generate random ranks. More specificly, for any person we need to generate a random permunation. In order to do this, we will generate n^2 real random values $X_{ij} \leftarrow U[0,1]$. Then for any fixed person i consider all X_{ij} except for X_{ii} and order numbers $1, 2, \dots, i-1, i+1, \dots, n$ with respect to order of X_{ij}. For example, for person 3, if $X_{31} = .43$, $X_{32} = .985$, $X_{34} = .1$, $X_{35} = .17$, then the ordered sequence will be $X_{34}, X_{35}, X_{31}, X_{32}$, which gives us permutation $4, 5, 1, 2$, and these numbers will be ranks for person 3. One can check that the resulting distribution of permutations is uniform.

Now let us call the matching to be standard (and denote as M_0), if each person i is paired with $i+\frac{n}{2}$. What is the probability that this matching is stable? Let us denote the event $A_{ij} = \{$ pair (i,j) is an instability $\}$. By definition, a matching M is stable if all pairs $(i,j) \notin M$ are not instabilities. Thus, $\{M_0$ is stable $\} = \bigcap\limits_{(i,j)\notin M_0} \bar{A_{ij}}$.

Now recall how we defined our ranks. Pair (i,j) is an instability if i likes j more than his partner $i+\frac{n}{2}$, and the same holds for j. This means that $A_{ij} = \{X_{ij} < X_{i,i+\frac{n}{2}}, X_{ji} < X_{j,j+\frac{n}{2}}\}$. Let's calculate the conditional probability: $Pr(A_{ij} | X_{\alpha,\alpha+\frac{n}{2}} = x_\alpha, \alpha = 1, \dots, n) = x_i x_j$ (here X_{ij} are random variables and x_i are numbers). Recall that $\{M_0$ is stable $\} = \bigcap\limits_{(i,j)\notin M_0} \bar{A_{ij}}$; therefore $Pr($ M_0 is stable$|X_{\alpha,\alpha+\frac{n}{2}} = x_\alpha, \alpha =$

$1, \ldots, n) = \prod_{(i,j) \notin M_0} (1 - x_i x_j)$. Now let's count all possible conditions: each $x_i \in [0, 1]$. Therefore, if $C = [0, 1]^n$ - n-dimentional cube, then

$$Pr(\text{ } M_0 \text{ is stable}) = \int_C \prod_{(i,j) \notin M_0} (1 - x_i x_j) d\bar{x}.$$

We need one more step: this is the probability only for M_0 to be stable. But due to symmetry, the same holds for any matching. Since there are $(n-1)!! = (n-1)(n-3) \cdot \ldots \cdot 5 \cdot 3 \cdot 1$ different matchings in a graph, the expected number of stable matchings

$$E[S_n] = (n-1)!! \int_C \prod_{(i,j) \notin M_0} (1 - x_i x_j) d\bar{x}.$$

With calculus it can be shown that $lim_{n \to \infty} E[S_n] = e^{\frac{1}{2}}$.

References

[1] Robert Irving. *An efficient algorithm for the stable roommates problem.*. 1985.

[2] Boris Pittel. *The average number of stable matchings.*. 1988.

[3] Boris Pittel. *The stable roommates problem with random preferences.*. 1993.